CANTOR SET IS COMPACT AND EQUAL TO SET OF CLUSTER PTS

note: in this proof, I will use the definition of the Cantor set that ross refrences in example 5 specifically reference [C61, 2.44] is rudin's definition of the Cantor set C.

$C_0 = [0,1]$, C_n is the union of 2^n intervals, each of length 3^{-n} and do not contain a segment of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$

We can conclude $C_{n+1} = C_n - \bigcup_{k=1}^{3^n}\left(\frac{3k-2}{3^{n+1}}, \frac{3k-1}{3^{n+1}}\right)$. Then $C = \bigcap_{n=1}^{\infty} C_n$

hypothesis: the Cantor Set $C = \bigcap_{n=1}^{\infty} C_n$, $C_0 = [0,1] \subset \mathbb{R}$

$$C_{n+1} = C_n - \bigcup_{k=1}^{3^n}\left(\frac{3k-2}{3^{n+1}}, \frac{3k-1}{3^{n+1}}\right)$$

Claim 1: the cantor set is compact.

Subclaim: each C_n is nonempty since $\frac{1}{3} \in C_n \ \forall n$

Basis: $\frac{1}{3} \in [0,1] \wedge \frac{1}{3} \in C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

inductive step: assume $\frac{1}{3} \in C_n$

$$\frac{1}{3} \in C_{n+1} \text{ is } \frac{1}{3} \notin \bigcup_{k=1}^{3^n}\left(\frac{3k-2}{3^{n+1}}, \frac{3k-2}{3^{n+1}}\right)$$

For the sake of contradiction, suppose $\frac{1}{3} \in \left(\frac{3k-2}{3^{n+1}}, \frac{3k-2}{3^{n+1}}\right)$

for some $k = 1, 2, \dots, 3^n$

then $\frac{3k-2}{3^{n+1}} < \frac{1}{3} < \frac{3k-1}{3^{n+1}}$

$3k-2 < 3^n < 3k-1$

but 3^n is an integer $\forall n$ and in $\mathbb{Z}$ $3k-2$ is the sucessor of $3k-1$, so there cannot be any integer between $3k-2$ and $3k-1$ $\forall k$

hence $\frac{1}{3} \notin \left(\frac{3k-2}{3^{n+1}}, \frac{3k-2}{3^{n+1}}\right)$ for any n and any k

$\Rightarrow \frac{1}{3} \in C_{n+1}$

by the principle of mathematical induction, $\frac{1}{3} \in C_n \ \forall n$

Hence $\frac{1}{3} \in C$

Subclaim 2: (C_n) is a sequence of decreasing sets

basis: $C_0 \supseteq C_1$ ✓

$C_1 \supseteq C_2$ since $[0,\frac{1}{3}]\cup[\frac{2}{3},1] \supseteq [0,\frac{1}{9}]\cup[\frac{2}{9},\frac{3}{9}]\cup[\frac{6}{9},\frac{7}{9}]\cup[\frac{8}{9},1]$ ✓

inductive step: assume $C_{n-1} \supseteq C_n$

$x \in C_{n+1}$

$$\Rightarrow x \in C_{n+1} + \bigcup_{k=1}^{3^n}\left(\frac{3k-2}{3^{n+1}},\frac{3k-1}{3^{n+1}}\right) = C_n - \bigcup_{k=1}^{3^n}\left(\frac{3k-2}{3^{n+1}},\frac{3k-1}{3^{n+1}}\right) + \bigcup_{k=1}^{3^n}\left(\frac{3k-2}{3^{n+1}},\frac{3k-1}{3^{n+1}}\right) = C_n$$

Hence $x \in C_{n+1} \Rightarrow x \in C_n$

$\Rightarrow C_{n+1} \subseteq C_n$

by the principle of mathematical induction, (C_n) is a sequence of decreasing sets in $\mathbb{R}$

This fact also implies each C_n is bounded since $C_n \in C_0 = [0,1]$ $\forall n$

Subclaim 3: the intersection of any set of closed intervals in $\mathbb{R}$ is closed

from discussion 13.7, since $\mathbb{R}$ is a topology the union of any number of open sets in $\mathbb{R}$ is open

That is, if A_n is a set of any size of open intervals in $\mathbb{R}$,

$\bigcup A_n$ is open,

Since the complement of an open set is closed,

$(\bigcup A_n)^c$ is closed

$\Rightarrow \bigcap A_n^c$ is closed, where each A_n^c is closed

Hence the intersection of any number of closed intervals is closed

Subclaim 4: each C_n is closed

Basis: $C_0 = [0,1]$ is ~~open~~ closed,

$C_1 = [0,\frac{1}{3}]\cup[\frac{2}{3},1]$ is closed by subclaim 3

inductive step: assume C_n is closed, let $M = \bigcup_{k=1}^{3^n}\left(\frac{3k-2}{3^{n+1}},\frac{3k-1}{3^{n+1}}\right)$

$C_{n+1} = C_n \setminus M = C_n \cap M^c$, M^c is closed since M is open,

by Subclaim 3, $C_n \cap M^c = C_{n+1}$ is closed

by the principal of mathematical induction, each C_n is closed $\forall n \in \mathbb{N}$

By Thrm 13.10 since (C_n) is a decreasing sequence of closed, bounded, nonempty sets,
$C = \bigcap_{n=0}^{\infty} C_n$ is closed, bounded, and nonempty

By 13.12 Heine-Borel Thm
$C \subseteq \mathbb{R}$ is compact since it is closed and bounded

hypothesis 2: Same as hypothesis 1

CLAIM 2: C is equal to its set of cluster points

Subclaim 2.1 : for $a, b, x, y \in \mathbb{R}$ s.t. $a < x < y < b$, $[a,b] - (x,y) = [a,x] \cup [y,b]$

first $p \in [a,b] \setminus (x,y) \Rightarrow p \notin (x,y)$
$\Rightarrow p \in [a,x]$ or $p \in [y,b]$
$\Rightarrow p \in [a,x] \cup [y,b]$

Now, $p \in [a,x] \cup [y,b] \Rightarrow p \in [a,b] \wedge p \notin (x,y) \Rightarrow p \in [a,b] \setminus (x,y)$

Hence $[a,b] \setminus (x,y) = [a,x] \cup [y,b]$

Subclam 2.2. each C_n is the union of disjoint closed intervals for $n \in \mathbb{N}$

basis: $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ ✓

now, assume the claim is true for C_n

Let I_n be the set of disjoint closed intervals makin up C_n

C_{n+1} is obtained by removing the open middle third of each I_n in C_n

by ~~claime~~ Subclaim 2.2, each interval in I_n equal to the union of two disjoint closed intervals.

Hence C_{n+1} is the union of closed disjoint closed intervals in $\mathbb{R}$

by the principal of mathematical induction, each C_n is the union of closed disjoint intervals in $\mathbb{R}$

Subclaim 2.3: C contains the boundary points of the closed disjoint intervals making up C_n for all $n \in \mathbb{N}$

Let I_n refer to the set of closed disjoint intervals making up C_n

Basis: C_1 contains the boundary points of C_0 ✓

induction: assume C_n contains the boundary points of all intervals in I_{n-1} making up C_{n-1}

Since each C_n is the union of disjoint closed intervals we obtain C_{n+1} by removing the open middle third of each of these intervals as we are consistently removing the open middle third, we never remove the boundary points of C_n

Hence C_{n+1} contains the boundary points of each interval in I_n making up C_n

by the principal of mathematical induction each C_n contains the boundary points of all intervals in each I_n

$\Rightarrow$ C contains all the boundary points of every interval making up C_n $\forall n \in \mathbb{N}$

Subclaim 2.4: $a \in C \Rightarrow a$ is a limit point of C

Suppose $a \in C$

Consider $B_r(a)$

For large enough n $\frac{1}{3^n} < r$

Since $a \in C \subseteq C_n$ and C_n is the union of several disjoint intervals, one of these intervals I_n contains a,

also $\text{diam}(I_n) = 3^{-n}$ so if $x \neq a$ is an end point of I_n

then $d(a,x) \leq \text{diam}(I_n) = 3^{-n} < r \Rightarrow x \in B_r(x)$

From our construction of the cantor set, all the endpoints of the I_n intervals are in C, so $x \in B_r(a) \cap C$

$\Rightarrow$ a is a limit point of C

Then,

Since C is closed and bounded, C contains all its limit points

Hence C is equal to its set of limit points

Subclaim 2.5: p is a cluster point in $C \Leftrightarrow p$ is a limit point in C

First I will show p is a cluster point $\Rightarrow p$ is a limit point in C

Since p is a cluster point, $B_r(p) \cap C$ $\forall r > 0$ contains at least one element since by defn it contains infinite elements. It follows that p is also a limit point of C.

Now I will show that p being a limit point of C implies p is a cluster point of C.

Since p is a limit point of C, $B_r(p) \cap C$ $\forall r > 0$ contains at least one element. Lets refer to this element as p'. Since all elements in C are limit points, $B_r(p') \cap C$ also contains at least one element. We can choose r small enough s.t. $B_r(p') \subseteq B_r(p)$. Then we can

continue this pattern infinetly to see $B_r(p) \cap C$ contains infinite elements.

Thus, p is also a cluster point.

it follows that p is a cluster point in C iff p is a Limit Point. C

Therefore the set of cluster points in is equal to the set of limit points in C

Hence C = set of all it's cluster points as desired ∎